Quadratic Backward Stochastic Differential Equations (BSDEs) Driven by a Continuous Martingale and Application to the Utility Maximization Problem

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Abstract

In this paper, we will study some quadratic Backward Stochastic Differential Equations (BSDEs) in a continuous filtration which arise naturally in the problem of utility maximization with constraints on the portfolio.
In the first part, we will show existence and uniqueness for those BSDEs. Then we will give an application to the utility maximization problem for three different cases: the exponential utility function, the power one and the logarithmic one.

Keywords:

Backward Stochastic Differential Equations (BSDE), continuous filtration, quadratic growth, utility maximization, incomplete market, constraint portfolio.

1 Introduction

1.1 Motivation

In this paper, we will study some quadratic BSDEs: these equations arise naturally in the utility maximization problem.
There is a long list of papers dealing with the classical problem of utility maximization and we will mention only a few of them, close to our setting.
The main interest comes from the existence of incomplete markets in which all contingent claims (or random variables depending on the information available at terminal time T) are not attainable. This explains the interest to introduce a new notion of optimality (and especially of optimal strategy).
To this aim, we consider a usual probability space (Ω, $\mathbb{F}$,($\mathcal{F}_t$), $\mathbb{P}$) equipped with a continuous and complete filtration $(\mathcal{F}_t)_{t\in[0,T]}$.
Then, we define the utility maximization problem by setting the value process
$V=(V(x_t))_{t\in[0,T]}$ as follows:

$$V(x_t)=\underset{\nu}{esssup}\,\mathbb{E}^{\mathcal{F}_t}\big(U(X_T^{\nu}\big)=\underset{\nu}{esssup}\,\mathbb{E}^{\mathcal{F}_t}(U(x_t+\int_t^T {\nu_s}'\frac{dS_s}{S_s})) \tag{1}$$

where x_t will be a fix $\mathcal{F}_t$-measurable random variable, S is the price process and the process: $X^\nu = x + \int(\nu^{'}.\frac{dS}{S})$ stands for the wealth process associated to the strategy ν.

The problem (1) is studied extensively in the literature, see [15] for a survey on this topic. The convex duality method is largely used for this type of problem, but this method requires the constraint on the portfolio to be convex.
Another method to solve this problem is to apply the BSDE technique see for example [7], [8] or recently [12]. We mention that in [7] the constraint on the portfolio is a convex cone, and that in [12], no constraint is imposed on the portfolio, but the authors study the problem in a general filtration. On the other hand in [8], the authors study the problem (1) in a Brownian filtration and, in particular, they prove the existence of optimal portfolio with a closed (but non necessarily convex) constraint on the portfolio. Because those authors work in a Brownian setting, the results on quadratic BSDEs are available (for this see [10]).

In the present article, we will study the problem (1) using the BSDE technique inspired by [8] but, since we will work on a continuous (but non Brownian) filtration, no results on quadratic BSDEs are available. Hence we begin by a study of the problem of existence and uniqueness for those quadratic BSDEs.
Then, in a second part, we will apply those results to find a construction of the utility value process $(V(x_t))_t$. We will compute for the exponential, power and logarithme utility functions the expression of this value process.

1.2 Theoretical background

In this part, we are going to introduce the type of the BSDE we will consider in the sequel. We consider as usual a probability space (Ω, F, $\mathbb{P}$) equipped with a continuous and complete filtration $\mathbb{F} = (\mathcal{F}_t)_t$ and with a continuous d-dimensional local martingale M .
In the sequel, all processes will be considered on [0, T] where T is a deterministic time (T is the horizon or maturity time in finance). Then, all local and $\mathbb{R}$-valued continuous martingales are supposed to be of the form: $K = Z^{'}.M + L$, where Z is a process taking its values in $\mathbb{R}^{d\times 1}$ and L is a continuous real valued martingale which is strongly orthogonal to M.
(that is to say that for each i: $< M^i, L > = 0$)
The stochastic integral w.r.t M of the $\mathbb{R}^d$ -valued process Z will be denoted as: $Z^{'}.M$.
We have moreover that each component $d < M^i, M^j >$ $(i, j \in [1, d]^2)$ of the quadratic variation of M is absolutely continuous with respect to $d\tilde{C}_s = d(\sum_i d < M^i >)$.

This is a simple consequence of Kunita-Watanabe's inequality for all continuous local martingales. In fact, using the result of Proposition 1.15 (chapter IV in Revuz-Yor, ([14])), we have the following inequality (for any i, j):

$$\int_0^t |d < M^i, M^j >_s| \leq \sqrt{< M^i >_t}.\sqrt{< M^j >_t} \leq \frac{1}{2}(< M^i >_t + < M^j >_t)$$

Since the process $\tilde{C}$ is in general unbounded, we set C as the bounded, real-valued and increasing process defined by: $C_t = \arctan(\tilde{C}_t)$.
It entails that: $dC_t = \frac{1}{1+\tilde{C}_t^2} d\tilde{C}_t$, and as a consequence, each component $d < M^i, M^j >$ of the quadratic variation process $d < M >$ is absolutely continuous w.r.t dC_t.

For each i, j, we have also existence of a random Radon Nikodym density $z^{i,j}$ such that: $d < M^i, M^j >_s = z^{i,j} dC_s$.
Finally thanks to the fact that $z = (z^{i,j})$ is a non negative and symmetric matrix, it implies that we can write: $d < M, M >_s = m_s^T m_s dC_s$, where m is a predictable process taking its values in $\mathbb{R}^{d\times d}$.The notation m^T stands for the transposed matrix. We will impose furthermore that the matrix $m_s^T m_s$ is invertible (for any s).
We are interested in finding a solution to the following BSDE:

$$(1.1) \begin{cases} dY_s = -F(s, Y_s, Z_s)dC_s - \frac{\beta}{2}.d < L >_s + Z_s^{'}.dM_s + dL_s \\ Y_T = B. \end{cases}$$

Moreover and in all the sequel in this paper, we will impose that the terminal condition B is bounded: we note here that the quadratic BSDEs with unbounded terminal value is studied by Briand and Hu in [3] and we hope to study this problem in a future publication.
A solution to such a BSDE is a triple of processes $(Y,\ Z,\ L)$ with $< L, M >= 0$ in the following space:
$S^\infty \times L^2(d < M > \times d\mathbb{P}) \times \mathcal{M}^2([0, T])$ equipped with the norms:
$|Y|_{S^\infty} = esssup(\sup_{0\le t\le T} |Y_t|)$,
$|Z|_{L^2(d<M>\times d\mathbb{P})} = \mathbb{E}(\int_0^T |m.Z_s|^2 dC_s)^{\frac{1}{2}}$,

$|L|_{\mathcal{M}^2([0,T])} = \mathbb{E}(< L >_T)^{\frac{1}{2}}$.

In the sequel, we will impose furthermore that we have the following conditions on F:

$$\exists\ \alpha \in L^1(dC_s),\ \alpha > 0\ \int_0^T \alpha_s dC_s \le a\ (a > 0) \text{ and: } b,\ \gamma > 0, \text{ such that:}$$

$$|F(s, y, z)| \le \alpha_s(1 + b|y|) + \frac{\gamma}{2}|mz|^2 \tag{H$_1$}$$

We suppose furthermore (without loss of generality) that: $\gamma \ge \beta$ (β has been introduced in the expression of the BSDE (1.1)), and $\gamma \ge b$.
To obtain the existence, we will, in a first step, impose the restrictive condition on F:

$$0 \le F(s, y, z) \le \alpha_s + \frac{\gamma}{2}|mz|^2 \tag{H$_1'$}$$

We impose here the same assumptions as for (H$_1$) on the parameter α and what import in this second assumption is that the lower bound is globally Lipschitz in y and z (this entails the existence of a minimal solution in the Brownian setting studied in [10] and refering to [3]).
In the preceding inequalities, the notation $|.|$ stands for the Euclidean norm on $\mathbb{R}^d$.
In the proof of existence, we will introduce a second BSDE of the following type:

$$(1.2) \begin{cases} dU_s = -g(s, U_s, V_s).dC_s + V_s^{'} dM_s + dN_s \\ U_T = e^{\beta.B} \end{cases}$$

We will show that we have a one to one correspondence between the solutions of these two BSDEs when defining $U = e^{\beta.Y}$ (exponential change), where $(Y,\ Z,\ L)$ is a solution of (1.1).

2 Results about quadratic BSDEs

In this section, we will begin by giving some a priori estimates on the norms of processes solving BSDE of the form (1.1) and (1.2): this will be of great interest to prove the results of existence and uniqueness.
Before giving these proofs, we state here the results we are able to obtain.
We state below the results of existence:

Theorem 1 *(i) Suppose that the generator g satisfies the assumption* ($H_1^{'}$)*, then the BSDE (1.2) has a solution* $(U,\,V,\,N)$ *in the space* $S^\infty \times L^2(d<M>\times d\mathbb{P})\times\mathcal{M}^2([0,T])$*, with:* $<M,N>=0$.

(ii) If the generator F satisfies the assumption (H_1) *, then there exists a solution* $(Y,\,Z,\,L)$ *in the space* $S^\infty \times L^2(d<M>\times d\mathbb{P})\times\mathcal{M}^2([0,T])$ *to the BSDE (1.1), with:* $<M,L>=0$.

Corollary 1 *Supposing now that g satisfies the following assumption:*

$$\exists\,\alpha\in L^1(dC_s),\ \alpha>0\ \int_0^T\alpha_s dC_s\le a\ (a>0)\ \textit{and: } C>0$$

$$-C(\alpha_s+|u|+|m.v|)\le g(s,u,v)\le\alpha_s+\frac{\gamma}{2}|m.v|^2 \tag{H$_2$}$$

BSDE (1.2) has at least one solution $(U,\,V,\,N)$ *in* $S^\infty \times L^2(d<M>\times d\mathbb{P})\times\mathcal{M}^2([0,T])$ *and such that:* $<M,N>=0$.

To prove a result of uniqueness, we need another assumption on the increments of the generator F:
we will impose furthermore the following conditions:
There exists a sequence of positive processes $(\lambda_P)_P$ verifying that:
$\exists\,\mu_P>0,\ \int_0^T\lambda_P dC_s\le\mu_P$, a sequence of positive constants $(C_P)_P$ and a positive process θ satisfying: $\exists\,l>0,\ \int_0^T\theta_s^2 dC_s\le l$, such that:

$$\forall\,z\in\mathbb{R}^d,\ \forall y^1,y^2,\ |y^1|,|y^2|\le P$$
$$(y^1-y^2).(F(s,y^1,z)-F(s,y^2,z))\le\lambda_P(s)|y^1-y^2|^2$$
$$\forall\,y,\ |y|\le P,\ \forall\,z^1,\ z^2\in\mathbb{R}^d,$$

$$|F(s,y,z^1)-F(s,y,z^2)|\le C_P(\theta_s+|m.z^1|+|m.z^2|).|m.(z^1-z^2)| \tag{H$_3$}$$

The second assumption will be useful to justify the uniform integrability of a stochastic exponential, this will require the use of the notion of martingale BMO:
we recall that M is a BMO martingale if there exists a constant c (c > 0) such that, for all stopping time τ of $\mathcal{F}$, we have:

$$\mathbb{E}^{\mathcal{F}_\tau}(<M>_T-<M>_\tau)\le c$$

Theorem 2 *Under the assumption (H_1) and the condition (H_3) on the generators and provided the terminal condition is bounded, the BSDEs of the form (1.1) (resp. of the form (1.2)) defined in the section 1.2 has a unique one solution (Y, Z, L) in $S^\infty \times L^2(d < M > \times d\mathbb{P}) \times \mathcal{M}^2([0,T])$ (resp. (U, V, N) in $S^\infty \times L^2(d < M > \times d\mathbb{P}) \times \mathcal{M}^2([0,T])$)*

2.1 A priori estimates

In this section, we will study under the assumption that the generator F satisfies the condition (H_1) or ($\mathrm{H}_1^{'}$).
One remark is that it suffices here to consider a BSDE of the form (1.1) with a generator satisfying (H_1) (we obtain a BSDE of the second form when: $\beta \equiv 0$).

Lemma 1 *Keeping the same notations as those mentioned in section 1.2, all triple (Y, Z, L) of processes solving the BSDE (1.1) with the process Y bounded ($\mathbb{P}$-almost surely and for all t) satisfies the following assertions : there exists some constants c and C depending only on the constants γ, a, b and $|B|_\infty$ (given in (H_1)) and a constant $C^{'}$ depending on the preceding constants plus the estimates of the norm of Y in S^∞ such that:*

$$\mathbb{P} - \textit{almost surely } \forall t,\ c \leq Y_t \leq C\ ; \tag{2}$$

$$\forall \tau\ (\tau\ \mathbb{F}\textit{stopping time})$$

$$\mathbb{E}^{\mathcal{F}_\tau}\Big(\int_\tau^T |m.Z_s|^2 dC_s + < L >_T - < L >_\tau\Big) \leq C^{'}. \tag{3}$$

Proof:
One first remark is that the second estimate will give us a bound of the BMO norms of the square integrable martingales $Z^{'}.M$ and L.
We suppose in the sequel that we are given a solution (Y, Z, L) of the BSDE (1.1) with a generator satisfying (H_1) and with the process Y bounded.

To prove the estimates given by (2), we introduce the processes U and V as follows: $U_t = e^{K.Y_t}$, $V_t = Ke^{K.Y_t} Z_t$.

It can be easily proved, by using Itô's formula, that this process U is solution of the following BSDE :

$$\begin{cases} dU_t = -g(t, U_t, V_t)dt + V_t^{'} dM_t + K.U_t dL_t - \frac{1}{2} K.U_t(\beta - K)d < L >_t \\ U_T = e^{K.B} \end{cases}$$

whose generator g is given by the expression:

$$g(s, u, v) = K.u\Big(F(s, \frac{\ln(u)}{K}, \frac{v}{K.u}) - \frac{K}{2}|m.\frac{v}{K.u}|^2\Big).\mathbf{1}_{u>0}.$$

As a simple consequence resulting from the expression of the generator, if we want to give an upper bound of g independent of $|m.v|^2$, it entails that we have to take: $\mathrm{K} \geq \gamma$. In the sequel, we fix: $\mathrm{K} = \gamma$ (we recall here that: $\gamma \geq |\beta|$).

Since F satisfies the assumption given by (H_1), it entails that we obtain the following control on g :

$$g(s,u,v) \leq \gamma.\alpha_s u.(1+\frac{b}{\gamma}.|\ln(u)|).\mathbf{1}_{u>0}. \tag{4}$$

We proceed hereafter with the same method as the one in Briand and Hu in [3], we will compare the process U to the solution of a differential equation. To this aim, fixing ω and setting: z = B(ω) (z is real), we consider the solution $\phi(z)$ of the following equation:

$$\phi_t(z) = e^{\gamma.z} + \int_t^T H(s,\phi_s)dC_s(\omega)$$

where H is given by:

$$H(x) = \gamma.\alpha_s.x(1+\frac{b}{\gamma}.\ln(x))\mathbf{1}_{x>1} + \gamma.\alpha_s\mathbf{1}_{x\leq 1}.$$

We recall here that: $\frac{b}{\gamma} \leq 1$.
It is also easy to check:

$$\forall\ u>0,\ \gamma.\alpha_s u.(1+\frac{b}{\gamma}.|\ln(u)|) \leq H(s,u). \tag{5}$$

As in [3], we remark that H(s, .) is locally Lipschitz and convex. But, contrary to their paper, we work here in a non Brownian setting (the random process $C = (C_s)$ (which is of finite variation) replaces the deterministic process (s)).
To give the expression of the solution ϕ, we have to discuss the sign of z, because the expression of H depends on whether or not the function ϕ is greater than one.
(i) if z $\geq$ 0:
It is the simplest case, since we can see easily that: t $\to \phi_t$ is a decreasing function with terminal value equal to $e^{\gamma.z}$ (which is greater than one in this case). The expression of ϕ is given by:

$$\phi_t(z) = exp(\gamma.\frac{e^{\int_t^T b.\alpha_u dC_u}-1}{b})exp(\gamma.z.e^{\int_t^T b.\alpha_u dC_u})$$

(ii) if z < 0 (then: $\phi_T(z) = e^{\gamma.z} < 1$):
(a) If $e^{\gamma.z} + \int_0^T \gamma.\alpha_u dC_u \leq 1$, then the solution is defined for all t by:

$$\phi_t = e^{\gamma.z} + \gamma.\int_t^T \alpha_u dC_u$$

(b) Otherwise: $\exists\ 0<S<T,\ e^{\gamma.z} + \gamma.\int_S^T \alpha_u dC_u = 1$,

$$\phi_t = \big(e^{\gamma.z} + \gamma.\int_t^T \alpha_u dC_u\big)\mathbf{1}_{]S,T]}(t) + exp(\gamma.\frac{e^{\int_t^S b.\alpha_u dC_u}-1}{b})\mathbf{1}_{[0,S]}(t)$$

Then, we introduce the adapted process Φ defined by:
$\forall t,\ \Phi_t = \mathbb{E}^{\mathcal{F}_t}(\phi_t(B))$.($\mathbb{E}^{\mathcal{F}_t}$ stands for the conditional expectation w.r.t $\mathcal{F}_t$)
Introducing the following martingale: $K_t = \mathbb{E}^{\mathcal{F}_t}\big(\phi_T(B) + \int_0^T \mathbb{E}^{\mathcal{F}_s}(H(s,\phi_s))dC_s\big)$,
we claim that Φ is a semi martingale (whose martingale part is K) which satisfies the following BSDE:

$$\Phi_t = e^{\gamma.B} + \int_t^T \mathbb{E}^{\mathcal{F}_s}H(\phi_s(B))dC_s - (K_T - K_t)$$

Thanks to the inequalities (4) and (5) and applying the same method as that in Briand and Hu in [3], we can conclude that a comparison result holds and that, for all t:
$U_t = e^{\gamma.Y_t} \leq \Phi_t$, or equivalently : $Y_t \leq \frac{1}{\gamma} ln(\mathbb{E}^{\mathcal{F}_t}(\phi_t(B)))$.
To obtain the lower bound of Y, it is enough to apply the same method to the process $-Y$: this is justified since the BSDE given by the parameters (F$^{'}$, -B, -β) (where F$^{'}$ is defined by: F$^{'}$(s, y, z) = - F(s, -y, -z)) and whose solution is (-Y,-Z,-L) is such that: the generator F$^{'}$ satisfies again the assumption H_1 with the same parameters and so, it implies that we have: $-Y_t \leq \frac{1}{\gamma}.ln(\mathbb{E}^{\mathcal{F}_t}(\phi_t(-B)))$.

We also obtain the estimates given by (2) by setting:

$$c = \underset{\omega}{\operatorname{essinf}} \inf_t -\frac{1}{\gamma} ln(\Phi_t(-B)) \text{ , and: } C = \underset{\omega}{\operatorname{esssup}} \sup_t \frac{1}{\gamma} ln(\Phi_t(B))$$

One important remark is that it is easy to show that it is possible to give estimates of the process Y in S$^{\infty}$ which are independent of the parameter γ: we recall that $\phi_t(z)$ is increasing w.r.t z, and so:

$$\phi_t(B) \leq \phi_t(|B|) = exp(\gamma.\frac{e^{\int_t^T b.\alpha_u dC_u} - 1}{b})exp(\gamma.|B|.e^{\int_t^T b.\alpha_u dC_u})$$

This entails that: $\mathbb{E}^{\mathcal{F}_t}(\phi_t(B)) \leq e^{\gamma.\frac{e^{b.a}-1}{b}}.e^{\gamma.|B|_\infty e^{b.a}}$.
Since we have both: B $\leq |B|$, and: -B $\leq |B|$, we obtain the same upper bound for $\mathbb{E}^{\mathcal{F}_t}(\phi_t(-B))$.
And, consequently, we can claim: $\forall s,\ |Y_s| \leq \frac{e^{b.a}-1}{b} + |B|_\infty e^{b.a}$, $\mathbb{P}-$ almost surely.

Then, to prove the estimates given by (3), we will apply Itô's formula to the process $\psi_K(Y + m_0)$ (K and m_0 are constants which will be precised later).
The expression of ψ_K is given by: $\psi_K(x) = \frac{e^{K.x}-1-K.x}{K^2}$.
The following properties will be useful in the sequel:

$$\begin{cases} \psi_K{}'(x) \geq 0 & x \geq 0. \\ -K.\psi_K^{'} + \psi_K^{''} = 1. \end{cases}$$

Moreover, we will use the fact that there exists a constant m_0 such that:
$\forall s \in [0, T],\ Y_s + m_0 \geq 0$ $\mathbb{P}$- almost surely.
Since Y is a bounded process, it suffices to choose: m_0 = - $|Y|_{S^\infty}$ (norm of the process in S$^{\infty}$).
Let τ be an arbitrary stopping time of $(\mathcal{F}_t)_{t\in[0,T]}$. Taking then the conditional expectation with respect to $\mathcal{F}_\tau$, it provides us with:

$$\begin{aligned} \psi_K(Y_\tau + m_0) - & \ \mathbb{E}^{\mathcal{F}_\tau}(\psi_K(Y_T + m_0)) \\ & = -\mathbb{E}^{\mathcal{F}_\tau}\left(\int_\tau^T \psi_K^{'}(Y_s + m_0)(-F(s, Y_s, Z_s)dC_s - \tfrac{\beta}{2}.d < L >_s)\right) \\ & \quad -\mathbb{E}^{\mathcal{F}_\tau}\left(\int_\tau^T \psi_K^{'}(Y_s + m_0)(Z_s^{'}dM_s + dL_s)\right) \\ & \quad -\mathbb{E}^{\mathcal{F}_\tau}\left(\int_\tau^T \frac{\psi_K^{''}}{2}(Y_s + m_0)|m.Z_s|^2 dC_s + d < L >_s)\right) \end{aligned}$$

Then, remembering the upper bound on the generator F given by the assumption (H_1) and after simple computations, we obtain:

$$\begin{aligned}\psi_K(Y_\tau+m_0)\ &-\mathbb{E}^{\mathcal{F}_\tau}\psi_K(Y_T+m_0)\\ &\leq \mathbb{E}^{\mathcal{F}_\tau}\int_\tau^T \psi_K^{'}(Y_s+m_0)(|\alpha_s|(1+b|Y|_{S^\infty})dC_s\\ &\quad+\mathbb{E}^{\mathcal{F}_\tau}\int_\tau^T(\tfrac{\beta}{2}\psi_K^{'}-\tfrac{1}{2}\psi_K^{''})(Y_s+m_0)d<L>_s\\ &\quad+\mathbb{E}^{\mathcal{F}_\tau}\int_\tau^T(\tfrac{\gamma}{2}\psi_K^{'}-\tfrac{1}{2}\psi_K^{''})(Y_s+m_0)|m.Z_s|^2dC_s\end{aligned}$$

We have easily that the terms of the left member are bounded (independently of the stopping time τ) and it is the same for the first term of the right member (thanks to the integrability assumption on α).
We put in the left member the two last terms of the second member of the preceding inequality, and then we claim that: fixing K such that: K = γ, and remembering that: $\gamma\geq\beta$, we have: $\forall x\geq 0, f_1(x)=\frac{1}{2}.(-\gamma.\psi_\gamma^{'}+\psi_\gamma^{''})(x)=\frac{1}{2}>0$, on the one hand, and: $\frac{1}{2}(-\beta.\psi_\gamma^{'}+\psi_\gamma^{''})(x)=f_1(x)+\frac{1}{2}(-\beta+\gamma)\psi_\gamma^{'}(x)\geq\frac{1}{2}$, on the other hand.

We use those inequalities for $x=Y_s+m_0$, quantity which is almost surely non negative. It implies that there exists a constant $C^{'}$ (depending only on the parameters a,b, γ, and $|B|_\infty$) and which is independent of the stopping time τ) such that:

$$\mathbb{E}^{\mathcal{F}_\tau}\left(\int_\tau^T|m.Z_s|^2dC_s+(<L>_T-<L>_\tau)\right)\leq C^{'}$$

2.2 Uniqueness for the BSDE (1.1)

Proof of Theorem 2:

We suppose that we are given two solutions (Y^1, Z^1, L^1) and (Y^2, Z^2, L^2) to BSDE (1.1) with Y^1 and Y^2 bounded. Let P be a constant such that, $\mathbb{P}$- almost surely: $|Y^1|\leq P$ (respectively $|Y^2|\leq P$).
We set: $Y^{1,2}=Y^1-Y^2$, $Z^{1,2}=Z^1-Z^2$ and: $L^{1,2}=L^1-L^2$.
The existence of such a constant P is justified by the estimates established in the preceding section under the assumptions (H_1) and (H_3) on the generator F.
To achieve it, we begin by applying Itô's formula to the non negative semi martingale ($\tilde{Y}^{1,2}$) defined as follows:

$\forall t,\ \tilde{Y}_t^{1,2}=e^{\int_0^t 2.\lambda_P(s)dC_s}|Y_t^{1,2}|^2$.
It gives us:

$$d(\tilde{Y}_s^{1,2})=2.\lambda_P(s)\tilde{Y}_s^{1,2}dC_s+e^{\int_0^s 2.\lambda_P dC_u}2.Y_s^{1,2}dY_s^{1,2}+\frac{1}{2}e^{\int_0^s 2.\lambda_P(u)dC_u}2d<Y^{1,2}>_s$$

We recall that we have:
$dY_s^{1,2}=-(F(s,Y_s^1,Z_s^1)-F(s,Y_s^2,Z_s^2))dC_s-\frac{\beta}{2}d(<L^1>_s-<L^2>_s)+dK_s$,
where K stands for the martingale part: $dK=\ (Z^{1,2})^{'}dM+dL^{1,2}$.

Then, taking the integral between t and T, we deduce:

$$\begin{aligned}\tilde{Y}_t^{1,2}-\tilde{Y}_T^{1,2} = &-\int_t^T 2.\lambda_P(s)\tilde{Y}_s^{1,2}dC_s\\ &+\int_t^T e^{\int_0^s 2.\lambda_P(u)dC_u}(2.Y_s^{1,2}(F(s,Y_s^1,Z_s^1)-F(s,Y_s^2,Z_s^2))dC_s)\\ &+\int_t^T e^{\int_0^s 2.\lambda_P(u)dC_u}2.Y_s^{1,2}\frac{\beta}{2}d(<L^{1,2},L^1+L^2>_s\\ &-\int_t^T e^{\int_0^s 2.\lambda_P(u)dC_u}2.Y_s^{1,2}\big((Z_s^{1,2})'.dM+dL^{1,2}\big)\\ &\underbrace{-\int_t^T e^{\int_0^s 2.\lambda_P(u)dC_u}\frac{1}{2}.2d<Y^{1,2}>_s}_{\leq 0}\end{aligned}$$

Considering the assumptions given by (H_1) and (H_3), we have:

$$2.Y_s^{1,2}(F(s,Y_s^1,Z_s^1)-F(s,Y_s^2,Z_s^2))\leq 2.\lambda_P(s)|Y_s^{1,2}|^2+2.Y_s^{1,2}.(m.\kappa_P)^T.(m.Z_s^{1,2})$$

when defining the process κ_P as follows:

$$\begin{cases}\kappa_P(s)=\frac{(F(s,Y_s^1,Z_s^1)-F(s,Y_s^2,Z_s^2)).(Z_s^{1,2})^T}{|m.(Z_s^{1,2})|^2} & \text{if } |m.(Z^{1,2})|\neq 0.\\ \kappa_P(s)=0 & \text{otherwise.}\end{cases}$$

Hence, we obtain:

$$\begin{aligned}\tilde{Y}_t^{1,2}\leq &\int_t^T 2.Y_s^{1,2}e^{\int_0^s 2.\lambda_P(u)dC_u}(m.\kappa_P).(m.Z_s^{1,2})dC_s\\ &+\int_t^T 2.Y_s^{1,2}e^{\int_0^s 2.\lambda_P(u)dC_u}\frac{\beta}{2}.(d<L^{1,2},L^1+L^2>_s)\\ &-\int_t^T 2.e^{\int_0^s 2.\lambda_P(u)dC_u}.Y_s^{1,2}(Z_s^{1,2})'dM-\int_t^T 2.e^{\int_0^s 2.\lambda_P(u)dC_u}Y_s^{1,2}dL^{1,2}\end{aligned}$$

Considering the following stochastic integrals:
on the one hand: $N=\left(2.e^{\int_0^t 2.\lambda_P(s)dC_s}Y^{1,2}Z^{1,2}\right)'.M$, and: $\bar{N}=\kappa_P'.M$, and on the other hand: $L=\left(2.Y^{1,2}e^{\int_0^t 2.\lambda_P(s)dC_s}L^{1,2}\right)$, and: $\bar{L}=\frac{\beta}{2}.(L^1+L^2)$

We define the new probability measure $\mathbb{Q}$ by: $d\mathbb{Q}=\mathcal{E}(\kappa_P'.M+\frac{\beta}{2}(L^1+L^2))d\mathbb{P}$. From Girsanov's theorem, it results that: $N+L-<N+L,\kappa_P'.M+\frac{\beta}{2}(L^1+L^2)>$ is a martingale under $\mathbb{Q}$.
In fact, the application of Girsanov's theorem is justified since, from (H_3), we deduce that:

$$|m.\kappa_P(s)|\leq C_N(\theta_s+|m.Z_s^1|+|m.Z_s^2|)$$

And, also: $\kappa_P'.M+\frac{\beta}{2}(L^1+L^2)$ is a BMO martingale, thanks to the estimate given by (3) and the assumption on the process θ given by (H_3). As a direct consequence (we refer here in particular to the results given by Kazamaki in [9]) : $\mathcal{E}(\kappa_P'.M+\frac{\beta}{2}(L^1+L^2)$ is a uniformly integrable martingale. Besides, we have:

$$N+L-<N+L,\kappa_P'.M+\frac{\beta}{2}(L^1+L^2)>=N-<N,\bar{N}>+L-<L,\bar{L}>$$

Remarking that the semi martingale $\tilde{Y}^{1,2}$ satisfies the following inequality:

$$\tilde{Y}_t^{1,2} \leq \int_t^T (d < N, \bar{N} >_s - dN_s) + \int_t^T d < L, \bar{L} >_s - dL_s$$

and taking the conditional expectation with respect to $\mathcal{F}_t$, we can conclude that : $\tilde{Y}^{1,2} \leq 0$ $\mathbb{Q}$ - almost surely (and also: $\mathbb{P}$ - almost surely, since the two measures are equivalent).
Finally, thanks to the symmetry of the problem: $Y^{1,2} \equiv 0$.
This achieves the proof of uniqueness.

2.3 Existence

Proof of Theorem 1:
This proof will be achieved in three main steps and following the method given by Kobylanski in [10].

2.3.1 Proof of Existence

In a first step, we will show that to solve BSDE (1.1) under (H$_1$), it suffices to consider the simple assumption (H$_4$) (in the following sense that the problem of existence of a solution is equivalent under this new assumption).
Then, in a second step, we introduce by doing a formal computation an intermediate BSDE of the form (1.2). We intend to establish a correspondence between the existence of a respective solution to BDSE (1.1) and to BSDE (1.2), this under the assumption that the generators associated with these two BSDEs satisfy (H$_4$). Finally, we will be able to justify the formal change of variable and express a solution to BSDE (1.1) in terms of a solution to BSDE (1.2) (provided it exists).
The third and last step consists simply in constructing a solution to BSDE (1.2) when its generator g satisfies (H$_4$).

Step 1: Truncation in y

We aim at using the a priori estimates we have proved in Section 2.1 so that we can relax the assumption on the generator and then obtain precise estimates for an intermediate BSDE.
In this step, we will show that it is possible to restrict ourselves to the assumption (H$_4$), instead of (H$_1$), on our generator F :

$$\exists \tilde{\alpha} \ \geq 0 \int_0^T \tilde{\alpha}_s dC_s \leq \tilde{a} \ (\tilde{a} > 0) \text{ , such that: } |F(s, y, z)| \leq \tilde{\alpha_s} + \frac{\gamma}{2}|m.z|^2 \tag{H$_4$}$$

Suppose now that we can construct the solution to BSDE (1.1) under (H$_4$) and let us explain how to construct the solution to BSDE (1.1) under (H$_1$). Defining:
K $= |c| + |C|$, where c and C are the constants given in (2) for a BSDE (1.1) under the assumption (H$_1$), we introduce the following BSDE:

$$\begin{cases} dY_s^K = -F^K(s, Y_s^K, Z_s^K)dC_s - \frac{\beta}{2} d < L^K >_s + (Z_s^K)' dM_s + dL_s^K, \\ Y_T^K = B. \end{cases}$$

where F is supposed to satisfy (H_1) and: $\mathrm{F}^K(s,y,z) = F(s,\rho_K(y),z)$. The truncation function ρ_K is defined as follows:

$$\rho_K(x) = \begin{cases} -K & \text{if: } x < -K \\ x & \text{if: } |x| \leq K \\ K & \text{if: } x > K \end{cases}$$

This entails that we have:

$$\forall y \in \mathbb{R},\ z \in \mathbb{R}^d,\ |F^K(s,y,z)| \leq \alpha_s(1 + b|\rho_K(y)|) + \frac{\gamma}{2}|m.z|^2$$

It is clear that, since: $|\rho_K(x)| \leq |x|$, F^K satisfies again (H_1) with the same parameters as F and, consequently, it implies that, for all solutions (Y^K, Z^K, L^K) to the BSDE characterized by the parameters (F^K, B, β), we have: $|Y^K| \leq K$. On the other hand, we have: F^K satisfies (H_2) with: $\forall s,\ \tilde{\alpha}_s = \alpha_s(1 + b.K)$.

Due to our assumption, there exists a solution (Y^K, Z^K, L^K) to the BSDE characterized by (F^K, B, β). Besides, since it is a solution of a BSDE whose generator satisfies (H_1): $|Y^K| \leq K$, which implies that:
$\mathrm{F}^K(s, Y_s^K, Z_s^K) = \mathrm{F}(s, Y_s^K, Z_s^K)$, and also:
(Y^K, Z^K, L^K) is a solution to the BSDE characterized by (F, B, β).

Step 2: an intermediate BSDE
To solve BSDE (1.1), we begin by setting formally: $U = e^{\beta.Y}$ and by supposing that we have a solution (Y, Z, L) to (1.1) .
Using Itô's formula, we show that this new process U is solution of a BSDE of the following form:

$$(1.2) \begin{cases} dU_s = -g(s, U_s, V_s)dC_s + (V_s)'.dM_s + dN_s \\ U_T = e^{\beta.B} \end{cases}$$

where we have defined the following processes: $V_s = \beta.U_s.Z_s$, and: $dN_s = \beta.U_s.dL_s$, so that the martingale part of U can be written: $V'.M + N$. The finite variation term is independent of d$< N >$ and its expression (in the differentiate form) is simply: $-g(s, U_s, V_s)dC_s$.
This BSDE is characterized by its generator g:

$$g(s,u,v) = \beta.u.F(s, \frac{\ln(u)}{\beta}, \frac{v}{\beta.u}) - \frac{1}{2.u}|m.v|^2.$$

We are willing to prove that if we can solve BSDE (1.2) given by (g, $e^{\beta.B}$), then we obtain a solution to BSDE (1.1) when setting:
$Y := \frac{\ln(U)}{\beta}$, $Z := \frac{V}{\beta.U}$, et: $dL := \frac{dN}{\beta.U}$

To this aim, we are eager to give precise estimates of the norm of the process U in S^∞ for all solutions (U, V, N) to the BSDE characterized by (g, $e^{\beta.B}$) . This is not achievable directly because of the singularity of this generator (in the variable u), and also we proceed hereafter by constructing a generator G (this by performing a truncation in u). Provided the generator F of BSDE (1.1) satisfies (H_4) and thanks to this truncation argument, it will lead us to a BSDE satisfying the assumption (H_1), for which we will be able to establish precise estimates and conclude that all solution to this new BSDE will

be solution to the BSDE given by (g, $e^{\beta.B}$) .
We define then G as follows (where c^1, c^2 are positive constants which will be precised later):

$$G(s,u,v) = \beta.\rho_{c^2}(u).F(s, \frac{\ln(u \vee c^1)}{\beta}, \frac{v}{\beta.(u \vee c^1)}) - \frac{1}{2.(u \vee c^1)}|m.v|^2,$$

the definition of the truncation function ρ_{c^2} is given in the first step.

As a consequence, remembering that F satisfies (H_4), we obtain:

$$|G(s,u,v)| \leq \beta.|u|(\tilde{\alpha_s} + \frac{\gamma.|m.v|^2}{2.|\beta.c^1|^2}) + \frac{|m.v|^2}{2.c^1} \leq \tilde{\beta}.u + \frac{\hat{\gamma}}{2}.|m.v|^2$$

where we have set: $\hat{\gamma} = \frac{\gamma.c^2}{|\beta|.|c^1|^2} + \frac{1}{c^1}$, $\tilde{\beta} = |\beta|\tilde{\alpha}.\,sign(u)$,.
Finally, we have proved that G satisfies, in particular, the assumption (H_1) with parameters: a: $= \tilde{a} =$ $|\tilde{\beta}|_{L^1(dC_s)}$, b: $= 1$, and: $\gamma := \hat{\gamma}$.
Thanks to the estimates established in Section 2.1, we have that for all solutions ($U^{c^1,\,c^2}$, $V^{c^1,\,c^2}$, $N^{c^1,\,c^2}$) with the process $U^{c^1,\,c^2}$ bounded:
$U^{c^1,\,c^2} \leq e^a - 1 + |e^{\beta.B}|_\infty e^a$, $\mathbb{P}$ - almost surely.(this upper bound is independent of the parameter γ and, as a consequence, independent of c^1 and c^2)
We define then: $c^2 = e^a - 1 + |e^{\beta.B}|_\infty e^a$.

It remains to see why for all solutions ($U^{c^1,\,c^2}$, $V^{c^1,\,c^2}$, $N^{c^1,\,c^2}$) the process U^{c^1,c^2} has a strictly positive lower bound.
Let (U, V, N) be a solution of the BSDE characterized by (G, $e^{\beta.B}$). Then, we defined the adapted process $\Psi(U)$ as follows:
$\forall t,\ \Psi(U_t) = e^{-\int_0^t \tilde{\beta}_s dC_s}.U_t$,
(Remembering that: $\tilde{\beta} = |\beta|\tilde{\alpha}.\,sign(U_s)$, this expression is well defined since $\tilde{\beta}$ is in $L^1(dC_s)$).
Then, applying Itô formula to $\Psi(U)$ and integrating between t and T , we claim that:

$$\begin{aligned}(\Psi(U_t) - \Psi(U_t)) = &\textstyle\int_t^T \big(e^{-\int_0^s \tilde{\beta} dC_u}(g(s,U_s,V_s) + \tilde{\beta}_s.U_s)\big)dC_s \\ &- \textstyle\int_t^T e^{-\int_0^s \tilde{\beta} dC_u}(V_s^{'}.dM_s + dN_s).\end{aligned}$$

Using that: $G(s,u,v) \geq -(\tilde{\beta}.u + \frac{\gamma}{2}|m.v|^2)$, we can finally claim: :

$$\Psi(U_t) - \Psi(U_T) \geq -\int_t^T \frac{\gamma}{2}.e^{-\int_0^s \tilde{\beta} dC_u}.|m.V_s|^2 dC_s - \int_t^T \big(e^{-\int_0^s \tilde{\beta} dC_u}(V^{'}.dM_s + dN_s)\big)$$

We then introduce the following change of probability measure by defining:
$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(-\frac{\gamma}{2}.V^{'}.M)$, the Girsanov's transform: $\tilde{M} = M + < \frac{\gamma}{2}.V^{'}.M, M >$ is justified thanks to the fact that, thanks to the a priori estimates (3), the process $V^{'}.M$ is a BMO martingale.
We can so rewrite the preceding inequality:

$$\Psi(U_t) - \Psi(U_T) \geq -\int_t^T \big(e^{-\int_0^s \tilde{\beta} dC_u}(V^{'}.d\tilde{M}_s + dN_s)\big)$$

Hence, taking under this new equivalent probability measure the expectation w.r.t $\mathcal{F}_t$, we obtain:

$$\Psi(U_t) \geq \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}(\Psi(U_T))$$

Consequently, we have: $U_t \geq \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}\big(\big(\inf U_T\big)^{-}.e^{-\int_t^T \tilde{\beta}_s dC_s}\big)$,
so: $U_t \geq e^{-\beta.|B|_\infty}.e^{-\tilde{a}}$, and we set: $\mathrm{c}^1 = e^{-\beta.|B|_\infty}.e^{-\tilde{a}}$.

For these choices of c^1, c^2, the generator G satisfies (H_1) and, consequently, if there exists a solution (U, V, N), this solution satisfies:

$$\mathbb{P}-\text{a.s}\ \ , c^1 \leq U \leq c^2.$$

We check easily that: $\forall s,\ G(s, U_s, V_s) = g(s, U_s, V_s)\ \mathbb{P}-$a.s.
This implies that: (U, V, N) is a solution of the BSDE characterized by (g, $\mathrm{e}^{\beta.B}$).
Since the process U is non negative and bounded, the formal computations are justified and we can set :
$Y := \frac{\ln(U)}{\beta}$, $Z := \frac{V}{\beta.U}$, et: $dL := \frac{dN}{\beta.U}$,
This provides us with a solution to BSDE (1.1).

Step 3: Approximation

In this step, we will prove the existence of a solution to (1.1) this under (H_1). The above two steps show that it is sufficient to prove the existence of a solution to (1.2) under (H_4). To achieve this, we aim at building a sequence of processes (U^n, V^n, N^n) which are solutions to BSDEs characterized by (g^n, B) and such that (U^n) is monotone. This will require the construction of a monotone sequence of Lipschitz functions (g^n) which will converge (locally uniformly) to g. Then, analogously to Kobylanski, we will establish the strong convergence of the sequences $((V^n)^{'}.M)$ and (N^n).
In the sequel, $||$ is an arbitrary norm (on $\mathbb{R}$ or $\mathbb{R}^d$).

To solve a BSDE of the form (1.2) characterized by the parameters (g, B), we first suppose that g satisfies ($\mathrm{H}_1^{'}$) and in particular that: $\mathrm{g} \geq 0$.
For each integer n, let us consider then the function:

$$g^n(s,u,v) = \inf_{u^{'},v^{'}} \Big(g(s,u^{'},v^{'}) + n|u-u^{'}| + n|m.(v-v^{'})| \Big)$$

To ensure the measurability, we take the infimum over $\mathbb{Q} \times \mathbb{Q}^d$.
Then g^n is well defined and globally Lipschitz continuous in the following sense:

$$|g^n(s,u^1,v^1) - g^n(s,u^2,v^2)| \leq n.|u^1-u^2| + n.|m.(v^1-v^2)| \tag{6}$$

Besides, since the sequence (g^n) is increasing and converges pointwise to g, Dini's theorem implies that the convergence is uniform over compact sets.

Moreover the fact that: $0 \leq g^n \leq g$, implies that:

$$\sup_n |g^n(s,0,0)| \leq \tilde{\alpha_s} \tag{7}$$

From those conditions (6) and (7) on the generators, we can show the existence, uniqueness and comparison theorems by classical arguments (see Pardoux and Peng [13] and El Karoui and Huang in [5]).

Thus, there exists a unique solution (U^n, V^n, N^n) to the BSDEs characterized by (g^n, B), and moreover the sequence (U^n) is increasing (this property results directly from the usual comparison theorem).
What is more is that thanks to the assumption (7) and the boundedness of the terminal condition and refering to classical results (see for example [2]), we can conclude that the sequence (U^m) is uniformly bounded in S^∞.
Then, it suffices to conclude the proof of Theorem 1 in the case when g ≥ 0, by applying the stability theorem we will state below in the following subsection.
To justify Corollary 1, it suffices to see as in [3] that under the assumption on the generator g the same construction of the sequence (g^n) holds.

In the general case (g is no longer supposed to be non negative), g satisfies (H_4), we proceed as in [3] by introducing the functions ($\mathrm{g}^{n,\,p}$) as follows:

$$\begin{aligned} g^{n,p}(s,u,v) = & ess \inf_{u^{'},v^{'}} \left(g^{+}(s,u^{'},v^{'}) + n|u-u^{'}| + n|m.(v-v^{'})|\right) \\ & - ess \inf_{u^{'},v^{'}} \left(g^{-}(s,u^{'},v^{'}) + p|u-u^{'}| + p|m.(v-v^{'})|\right) \end{aligned}$$

To obtain one solution to the BSDE we are interested in, it will be necessary to proceed with two successive passages to the limit: the solution U to the BSDE characterized by (g, B) will be equal to: $U = \lim_n \nearrow (\lim_p \searrow U^{n,p})$.
It is the same justification that holds true for these two passages to the limit, so, without any restriction, we will suppose in all the sequel that g satisfies the assumption given by ($\mathrm{H}_1^{'}$).

Firstly, thanks to the fact that (U^n) is an increasing sequence, then we can define: $\tilde{U} = \lim_n \nearrow (U^n)$.
Besides, since for all n the generator g^n satisfies (H_4), and remembering the a priori estimates (3), we have that:

$$\exists\, C^{'} > 0,\ \sup_{n\geq 0} \left(\mathbb{E}\Big(\int_0^T |m.V^n|^2 dC_s\Big) + \mathbb{E}(|N_T^n|^2) \right) \leq C^{'}$$

Hence, there exists a subsequence such that: $V^n \xrightarrow{w} \tilde{V}$ (in $\mathrm{L}^2(d < M > \times d\mathbb{P})$) and: $N_T^n \xrightarrow{w} \tilde{N}_T$ in $\mathrm{L}^2(\mathcal{F}_T)$. This implies also that: $N_t^n \xrightarrow{w} \tilde{N}_t$, which is defined as follows: $\tilde{N}_t = \mathbb{E}^{\mathcal{F}_t}(\tilde{N}_T)$.
However it is not sufficient to justify the passage to the limit in the BSDE characterized by (g^n, B): in fact, we need to have the strong convergence eventually along a subsequence to $(\tilde{U}, \tilde{V}, \tilde{N}\ )$.
We postpone the proof of this result linked to a lemma intitled "monotone stability" to the following subsection.

To achieve the proof of existence, it remains to justify the passage to the limit in the equation:

$$U_t^n = U_T^n + \int_t^T g^n(s, U_s^n, V_s^n) dC_s + \int_t^T (V_s^n)^{'}.dM_s + N_T^n - N_t^n$$

It is necessary to check the following assertions :
(i) $\int_0^t (V_s^n)' dM_s \xrightarrow{n\to\infty} \int_0^t (\tilde{V}_s)' dM_s$ $\mathbb{P}$-almost surely and for all t .
(ii) $N_t^n \xrightarrow{n\to\infty} \tilde{N}$ ($\mathbb{P}$-almost surely and for all t).
(iii) $\int_0^t g^n(s, U_s^n, V_s^n) dC_s \xrightarrow{m\to\infty} \int_0^t g(s, \tilde{U}_s, \tilde{V}_s) dC_s$

Assumptions (i) and (ii) are easily obtain from the strong convergence in L^2 of the sequences($(V^n)'.M$) and (N^n), since we can suppose that the convergence is achieved $\mathbb{P}$-almost surely (taking a subsequence if necessary) .
Finally, the assumption (iii) is a consequence of Lebesgue's theorem and using the same method as Kobylanski, we can claim that: $\sup_n(V^n)$ and $\sup_n(N^n)$ are square integrable, this justifies the domination.

As a consequence, this passage to the limit allows us to state that the triple of processes $(\tilde{U}, \tilde{V}, \tilde{N})$ is a solution of (1.2).

By setting: $Y = \frac{ln(\tilde{U})}{\beta}$, $Z = \frac{\tilde{V}}{\beta.\tilde{U}}$, and the martingale measure: $dL = \frac{d\tilde{N}}{\beta.\tilde{U}}$,
we obtain a solution to BSDE (1.1). □

2.3.2 Monotone stability

Lemma 2 *Considering the BSDE (1.2) and using the same notations as in the preceding section, if the sequence $(g^n)_n$ is such that:*
-For all s, the sequence of functions $((u,v) \to g^n(s,u,v))$ converges locally uniformly on $\mathbb{R} \times \mathbb{R}^d$ to g $((u, v) \to g(s, u, v))$.
- For all n, g^n satisfies the assumption (H_4):

$$\exists\, \tilde{\alpha} \in L^1(dC_s),\ \tilde{\alpha} \geq 0$$
$$|g^n(s,u,v)| \leq \tilde{\alpha}_s + \frac{\gamma}{2}|mv|^2$$

- the sequence (U^n) is increasing.
If, besides, we have existence of solutions (U^n, V^n, N^n) to those BSDEs given by the parameters (g^n, B), B being a bounded $\mathcal{F}_T$- measurable random variable, then:
The sequence (U^n, V^n, N^n) converges in the product space $S^\infty \times L^2(d<M> \times d\mathbb{P}) \times \mathcal{M}([0,T])$ to the triple $(\tilde{U}, \tilde{V}, \tilde{N})$, which is solution to BSDE (1.2).

Proof:

Following the same method as the one used by Kobylanski in [10], we will prove the strong convergence of the sequences $((V^n)'.M)_n$ and $(N^n)_n$ to $\tilde{V}'.M$ and $\tilde{N}$ (this will require the a priori estimates established in the section 2.1).

We write Itô's formula for the non negative semi martingale: $\Phi_K(U^n - U^p)$
(n $\geq$ p), where Φ_K is given by:

$$\Phi_K(x) = \frac{e^{K.x} - K.x - 1}{K^2}$$

K will be determined later.
This is a C^2 function which satisfies on the one hand, $\Phi_K \geq 0$, $\Phi_K(0) = 0$, and on the

other hand, $\Phi_K^{'}(x) \geq 0$, if x ≥ 0, and: $\Phi_K^{''} - K.\Phi_K^{'} = 1$.

Taking the integral between 0 and T and then, the expectation:

$\mathbb{E}\Phi_K(U_0^n - U_0^p) - \mathbb{E}\Phi_K(U_T^n - U_T^p) =$

$$\mathbb{E}\int_0^T (\Phi_K^{'}(U_s^n - U_s^p))(g^n(s, U_s^n, V_s^n) - (g^p(s, U_s^p, V_s^p))dC_s$$

$$-\mathbb{E}\int_0^T \frac{\Phi_K^{''}}{2}(U_s^n - U_s^p)|m.(V_s^n - V_s^p)|^2 dC_s$$

$$-\mathbb{E}\int_0^T \frac{\Phi_K^{''}}{2}(U_s^n - U_s^p)d < N^n - N^p >_s$$

We now aim at controlling the increments of the generators, so that we make appear the norm of the stochastic integral $(V^n - V^p)^{'}.M$ before putting it in the left-hand side and passing to the liminf when p $\rightarrow \infty$.
Using the fact that the generators g^n and g^p satisfy the assumption (H$_4$), we deduce:

$$|g^n(s, U_s^n, V_s^n) \ - g^p(s, U_s^p, V_s^p)|$$

$$\leq 2.\tilde{\alpha}_s + \frac{\gamma}{2}.|m.(V_s^n)|^2 + \frac{\gamma}{2}.|m.(V_s^p)|^2$$

$$\leq 2.\tilde{\alpha}_s + \frac{3.\gamma}{2}.\big(|m.(V_s^n - V_s^p)|^2 + |m.(V_s^p - \tilde{V}_s)|^2 + |m.\tilde{V}_s|^2\big)$$
$$+\gamma.\big(|m.(V_s^p - \tilde{V}_s)|^2 + |m.\tilde{V}_s|^2\big)$$

$$\leq 2.\tilde{\alpha}_s + \frac{3.\gamma}{2}.\big(|m.(V_s^n - V_s^p)|^2\big) + \frac{5.\gamma}{2}.\big(|m.(V_s^p - \tilde{V}_s)|^2 + |m.\tilde{V}_s|^2\big)$$

The two last inequalities result simply from the convexity of z $\rightarrow z^2$.

Putting in the left-hand side all the terms containing $(V^n - V^p)$, we obtain the following inequality (**):

$$\mathbb{E}\Phi_K(U_0^n - U_0^p) + \mathbb{E}\int_0^T \frac{\Phi_K^{''}}{2}(U_s^n - U_s^p)d < N^n - N^p >_s$$
$$+\mathbb{E}\int_0^T ((\frac{\Phi_K^{''}}{2} - \frac{3.\gamma}{2}.\Phi_K^{'})(U_s^n - U_s^p)|m.(V_s^n - V_s^p)|^2 dC_s)$$
$$\leq \mathbb{E}\int_0^T \Phi_K^{'}(U_s^n - U_s^p)\left(2.\tilde{\alpha_s} + \frac{5.\gamma}{2}|m.(V_s^p - \tilde{V}_s)|^2 + |m.\tilde{V}_s|^2\right) dC_s \quad (**)$$

Then, we fix a value for the parameter K such that:

$$\Phi_K^{''} - ((5+3).\gamma)\Phi_K^{'} \geq 1 \tag{8}$$

Consequently, fixing : $K = 8.\gamma$, this condition is trivially satisfied.
This last inequality (8) entails the strict positivity of the last term of the left-hand side. Besides, thanks to the weak convergence of a subsequence of (V^n) to $\tilde{V}$ (and similarly of

(N^n) to $\tilde{N}$), we have on the one hand:

$$\liminf_{n\to\infty} \mathbb{E}\int_0^T ((\frac{\Phi_K^{''}}{2} - \frac{3.\gamma}{2}.\Phi_K^{'})(U_s^n - U_s^p)|m.(V_s^n - V_s^p)|^2)dC_s \geq \qquad (9)$$
$$\mathbb{E}\int_0^T ((\frac{\Phi_K^{''}}{2} - \frac{3.\gamma}{2}.\Phi_K^{'})(\tilde{U}_s - U_s^p)(|m.(\tilde{V}_s - V_s^p)|^2)dC_s)$$

And, on the other hand:

$$\liminf_{n\to\infty} \mathbb{E}\int_0^T (\frac{\Phi_K^{''}}{2}.(U_s^n - U_s^p)d < N^n - N^p >_s \geq \qquad (10)$$
$$\mathbb{E}\int_0^T (\frac{\Phi_K^{''}}{2}.(\tilde{U}_s - U_s^p)d < \tilde{N} - N^p >_s)$$

The passage to the limit when n $\to \infty$ in the right-hand side results from an application of Lebesgue's theorem.
On the one hand, we have almost sure convergence of (U^n) to the process $\tilde{U}$.
On the other hand, the following quantity:
$\Phi_K^{'}(U_s^n - U_s^p)(|m.(\tilde{V}_s - V_s^p)|^2 + |m.\tilde{V}_s|^2 + \bar{C}_s)$ is integrable with respect to dC_s, because it is the product of a bounded process and a sum of integrable processes .

Passing to the limit in (**) when n goes to ∞ and using the inequalities (9) and (10), it gives us:

$$\begin{aligned}
&\mathbb{E}\Phi_K(\tilde{U}_0 - U_0^p) + \mathbb{E}\int_0^T \tfrac{\Phi_K^{''}}{2}(\tilde{U}_s - U_s^p)d < \tilde{N} - N^p >_s \\
&+\mathbb{E}\int_0^T ((\tfrac{\Phi_K^{''}}{2} - \tfrac{3.\gamma}{2}.\Phi_K^{'})(\tilde{U}_s - U_s^p)|m.(\tilde{V}_s - V_s^p)|^2 dC_s) \\
&\quad \leq \mathbb{E}\int_0^T \left(\Phi_K^{'}(\tilde{U}_s - U_s^p)(\tfrac{5.\gamma}{2}.|m.(\tilde{V}_s - V_s^p)|^2)dC_s\right) \\
&\quad +\mathbb{E}\int_0^T \left(\Phi_K^{'}(\tilde{U}_s - U_s^p)(2.\tilde{\alpha}_s + \tfrac{5.\gamma}{2}.|m.\tilde{V}_s|^2)dC_s\right)
\end{aligned}$$

To justify the passage to the limit when p $\to \infty$, it remains to put in the left-hand side the terms containing the following quantity $(|m.(\tilde{V} - V^p)|^2)$.
Remembering the properties of Φ_K and the condition (8) on K, the limit of the left-hand side is strictly larger than:

$$\mathbb{E}\left(\int_0^T .|m.(\tilde{V}_s - V_s^p)|^2 dC_s + < \tilde{N}_s - N_s^p >_T\right)$$

Concerning the right-hand side, it results from a simple the application of Lebesgue's theorem.
Finally we can conclude:

$$\liminf_{p} \mathbb{E}\left(\int_0^T |m.(\tilde{V}_s - V_s^p)|^2 dC_s + < \tilde{N}_s - N_s^p >_T\right) \leq 0$$

This can be expressed under the form:
V^p.M converges to $\tilde{V}.M$ in L$^2(d < M > \times d\mathbb{P})$.

N^p converges to $\tilde{N}$ in $\mathcal{M}^2([0,T])$.

3 Applications to finance

3.1 The case of the exponential utility

One important interest of this theoretical work is the link between the solution of BSDEs with quadratic growth and some problems arising from Mathematical Finance: we will in this section focus our attention on one particular problem dealing with the notion of utility value process as regards to the exponential utility of a portfolio.
We begin here by summing up the main assumptions about the model in our case of a general continuous filtration (we refer here to Mania et Schweizer in [12]).
As before, we are given a probability space and a continuous filtration $\mathbb{F}$.
We set: S= (S^i_t) the semi martingale which takes its values in $\mathbb{R}^d$ and which represents the discounted prices of d risky assets. The evolution of the price process S in our continuous setting is given by the following equation:

$$\frac{dS_s}{S_s} = dM_s + dA_s, \quad \text{with: } dA = d < M > .\lambda$$

(this expression is justified for example in [4] and in the particular context in the recent article [1]).
We recall here that M is a local continuous martingale of the filtration whose quadratic variation $d < M >$ can be written as explained in section 1.2 under the following form: $d < M >_s = m_s^T.m_s dC_s$.
A is a continuous process with bounded variation : we will suppose in the sequel that λ is a $\mathbb{R}^d$-valued process which is besides almost surely bounded and which satisfies:

$$\exists a > 0, \quad \int_0^T \lambda_s^T d < M >_s \lambda_s = \int_0^T |m.\lambda|^2 dC_s \leq a \qquad (H_3)$$

Furthermore, this expression for S provides a justification for the no arbitrage condition.
We define then the notion of a portfolio associated to a strategy ν:

Definition 1 *A $\mathbb{R}^d$-valued process ν which is predictable with respect to the filtration $\mathbb{F}$ is called trading strategy if the following stochastic integral: $\int \nu' \frac{dS}{S} = \int \sum_i \frac{\nu^i}{S^i} dS^i$ is well defined .*
Each component ν_i of the trading strategy corresponds to the amount of money invested in the i^{th} asset.
The process X^ν given below is called wealth process of an agent having the strategy ν and x represents the initial wealth:

$$\forall t \in [0,T] , \quad X_t^\nu = x + \int_0^t {\nu_s}' \frac{dS_s}{S_s} \qquad (11)$$

We suppose besides that we have an incomplete financial market, that is to say that all contingent claims (i.e. square integrable variables with respect to $\mathcal{F}_T$) are not attainable. A contingent claim B is attainable provided it exists a strategy ν (and also a process X^ν) such that: B = X_T^ν, where T represents the horizon (or maturity time) which will be a deterministic time in this work.
We then introduce the utility value process at time t, $V_t^B(x_t)$: it is an $\mathcal{F}_t$ random variable defined by:

$$V_t^B(x_t) = ess\sup_{\nu\in\mathcal{A}} \mathbb{E}^{\mathcal{F}_t}(U_\alpha(X_T^\nu - B))$$

where U$_\alpha$ is given by: $U_\alpha(x) = -exp(-\alpha.x)$, it stands for the utility function, $\mathcal{A}$ stands for the set of admissible strategies (in this expression of V_t, the strategy ν is viewed as a process), and the set C represents the set of constraints: it is a subset of $\mathbb{R}^d$ where all strategies take their values (i.e. for all t and $\mathbb{P}$ almost surely, $\nu_t(\omega)$ is in C , C $\subset \mathbb{R}^d$). We impose besides that: 0 is in C.
Then, before explaining how to solve this problem, let us introduce the notion of admissible strategy in our context:

Definition 2 *Let C be a closed (and non necessarily convex set) in* $\mathbb{R}^d$ *. The set* $\mathcal{A}$ *of admissible strategies consists of all d-dimensional predictable processes* $\nu = (\nu_t)_{t\in[0,T]}$ *satisfying:* $\mathbb{E}(\int_0^T |m.\nu|^2 dC_s) < \infty$*, as well as the uniform integrability of the family:*

$$\{exp(-\alpha.X_\tau^\nu),\ \tau \textit{ stopping time with values in } [0,T]\}$$

We aim here at giving an expression of the value process by a dynamical method: this approach will require the theoretical study we have made on quadratic BSDEs. The method applied here is the same as the one used in [8].
To achieve this, we introduce for all strategies ν the process R$^\nu$ by setting :
$\forall s$, R$_s^\nu = U_\alpha(X_s^\nu - Y_s)$.

We search to construct Y such that R$^\nu$ satisfies:
(i) R$_T^\nu$ = -exp(-$\alpha(X_T^\nu - F)$), for all strategies ν .
(ii) R$_t^\nu$ = R$_t$ = U$_\alpha(x_t - Y_t)$ (where x_t is a fix $\mathcal{F}_t$ random variable and such that:
for all strategy ν, $X_t^\nu = x_t$).
(iii) R$^\nu$ is a supermartingale for any strategies and there exists ν^* such that R$^{\nu^*}$ is a martingale.

Besides, this method allows us to give a positive answer to the existence of an optimal strategy.

3.2 A dynamical way to solve the problem:

In this section, we will show that the process $Y = (Y_t)$ is solution of a BSDE with quadratic growth of the same type as (1.1), whose parameters are β and the generator: F = F(s, z) :

$$\begin{cases} dY_s = -F(s, Z_s)dC_s - \frac{\beta}{2}.d < L >_s \\ \qquad\quad +Z_s^{'}.dM_s + dL_s \\ Y_T = B \end{cases}$$

Lemma 3 *Keeping the same notations, the process Y (such that the family of processes (R^ν) satisfies the assumptions (i), (ii) and (iii) introduced when describing the problem in the exponential case) is solution of a BSDE with quadratic growth of the form (1.1) with the following parameters: β is given by: $\beta = \alpha$ (corresponding to the risk aversion parameter), and the expression of F is:*

$$F(s,z) = \inf_{\nu \in C} \big(\frac{\alpha}{2}|m(\nu - (z + \frac{\lambda_s}{\alpha}))|^2\big)$$
$$-(m.z)^T.(m.\lambda_s) - \frac{1}{2.\alpha}|m.\lambda_s|^2, \;\; \mathbb{P} - a.s \tag{12}$$

The expression of the value process at time t is a consequence of the dynamical principle and it is given by:

$$V_t^B(x_t) = esssup_{\nu} \mathbb{E}^{\mathcal{F}_t}\big(U_\alpha(X_T^\nu - B)\big) = U_\alpha(x_t - Y_t)$$

Besides, there exists an optimal strategy ν^* which is defined almost surely for all s by:

$$\nu_s^* \in arg\min_{\nu \in C} |m.(\nu - (Z_s + \frac{\lambda_s}{\alpha}))|^2$$

There exists at least a measurable way to construct ν^* (but a priori it is not necessarily unique).

Before justifying how we can find the expression of the generator, we can remark that the expression of F given in Proposition 3 satisfies the assumption (H_1) with: b $\equiv$ 0 (F is independent of y).
In fact, we have the following lower bound for F:

$$F(s,z) \geq -(m.z)^T.(m.\lambda_s) - \frac{1}{2.\alpha}|m.\lambda_s|^2 \geq -|m.z|.|m.\lambda_s| - \frac{1}{2.\alpha}|m.\lambda_s|^2$$

and, we obtain:

$$F(s,z) \geq -\big(\frac{\alpha}{2}.|m.z|^2 + \frac{1}{\alpha}|m.\lambda_s|^2\big) \tag{13}$$

Then, if we set: $\forall s,\; \alpha_s = \frac{1}{\alpha}|m.\lambda_s|^2$, we can claim: α is in $L^1(dC_s)$, and: $\int_0^T \alpha_s dC_s \leq \tilde{a}$, with: $\tilde{a} > 0$ thanks to the assumption (H_3) made on the process λ.
The upper bound is obtained by remarking that: 0 is in C, this entails that:

$$F(s,z) \leq \frac{\alpha}{2}|m.z|^2$$

We now aim to check that the condition for uniqueness given by (H_3) is satisfied.
For all z^1, z^2 in $\mathbb{R}^d$, we have:

$$\begin{aligned}|F(s,z^1) - F(s,z^2)| &\leq |\tfrac{\alpha}{2}(dist^2(m.(z^1 + \tfrac{\lambda}{\alpha}), m.C) - dist^2(m.(z^2 + \tfrac{\lambda}{\alpha}), m.C)\,| \\ &\quad + | - (m.z^1)^T(m.\lambda) + (m.z^2)^T(m.\lambda)| \\ &\leq |\tfrac{\alpha}{2}.|m.(z^1 - z^2)|.\big(m.(|z^1| + |z^2| + 2.|\tfrac{\lambda}{\alpha}|)|\big) + |m.(z^1 - z^2)|.|m.\lambda| \\ &\leq \tfrac{\alpha}{2}.|m.(z^1 - z^2)|.(\theta_s + |m.z^1| + |m.z^2|)\end{aligned}$$

Consequently, the assumption (H_3) is satisfied with: $C_P = \frac{\alpha}{2}$ and: $\theta = m.\frac{4.|\lambda|}{\alpha}$ (the assumption holds true for this process thanks to (H_3)).
According to the preceding section, this entails that we have existence and uniqueness for the BSDE characterized by its terminal condition B and its generator F.

Proof of Lemma 3
We recall first the two following expressions:

$$X_t^\nu = x + \int_0^t \nu_s^{'} dM_s + \int_0^t (m.\nu_s)^T.(m.\lambda_s)dC_s$$

and:

$$Y_t = Y_0 - \int_0^t F(s, Z_s)dC_s - \frac{\beta}{2} < L >_t + \int_0^t Z_s^{'} dM_s + L_t$$

So it entails that:

$$\begin{aligned} X_t^\nu - Y_t = & (x - Y_0) + \textstyle\int_0^t (\nu_s - Z_s)^{'} dM_s - L_t \\ & + \textstyle\int_0^t F(s, Z_s)dC_s + \frac{\beta}{2} < L >_t + \int_0^t (m.\nu_s)^T (m.\lambda_s)dC_s \end{aligned}$$

Using then the following two simple equalities (by remembering the definition of the stochastic exponential $\mathcal{E}(M)$ of a continuous martingale M):

$$\begin{aligned} & exp(-\alpha(\textstyle\int_0^{\cdot} (\nu_s - Z_s)^{'} dM_s)) = \\ & \quad \mathcal{E}(-\alpha.(\textstyle\int_0^{\cdot} (\nu_s - Z_s)^{'} dM_s)).exp(\frac{\alpha^2}{2} \int_0^{\cdot} |m.(\nu_s - Z_s)|^2 dC_s) \end{aligned}$$

and: $exp(\alpha.L_{.}) = \mathcal{E}(\alpha.L).exp(\frac{\alpha^2}{2} < L >)$,
we deduce that the process R^ν can be written as follows:

$$R_s^\nu = -exp(-\alpha.(x - Y_0))\mathcal{E}(-\alpha(\nu - Z)^{'}.M)\mathcal{E}(\alpha.L).e^{A^\nu}$$

Besides, since M and L are strongly orthogonal, it implies that:
$\mathcal{E}(-\alpha(\nu - Z)^{'}.M)\mathcal{E}(\alpha.L) = \mathcal{E}(-\alpha(\nu - Z)^{'}.M + \alpha.L)$.
Writing that: $A^\nu = \int_0^{\cdot} dA^\nu$, we have that:

$$\begin{aligned} dA_s^\nu = & \big(- \alpha.F(s, Z) - \alpha.(m.\nu)^T.(m.\lambda) + \tfrac{\alpha^2}{2}|m.(\nu - Z)|^2\big)dC_s \\ & + (\tfrac{\alpha^2 - \alpha.\beta}{2})d < L >_s . \end{aligned}$$

It appears then that R^ν is the product of a local martingale and a finite variation process which has to be a non decreasing process:
A sufficient condition for R^ν to be a super martingale is: $A^\nu \geq 0$ (it is a martingale for ν^* such that: $A^{\nu^*} \geq 0$).
Consequently, since the stochastic exponential is a positive local martingale, there exists a sequence of stopping time (τ_n) such that: $(R_{t\wedge\tau_n}^\nu)$ is a super martingale (for each ν).As a consequence:

$$\forall\ A \in \mathcal{F}_s \quad \mathbb{E}^{\mathcal{F}_s}\big(R_{t\wedge\tau_n}.\mathbf{1}_A\big) \leq R_{s\wedge\tau_n}.\mathbf{1}_A$$

Passing to the limit, we deduce that: $\mathbb{E}^{\mathcal{F}_s}\big(R_t.\mathbf{1}_A\big) \leq R_s.\mathbf{1}_A$, from the uniform integrability of ($R_{t\wedge\tau_n}$) which is a direct consequence of both the definition of admissibility and the boundedness of the process Y.

Reformulating the condition: $e^{A^\nu} \geq 1$, we obtain:

$$\begin{cases} -\alpha.\frac{\beta}{2}d < L >_s + \frac{\alpha^2}{2}d < L >_s = 0 \Rightarrow \beta = \alpha \\ -\alpha.(F(s, Z_s) + (m_s\nu_s)^T m_s\lambda_s) + \frac{\alpha^2}{2}|m_s(\nu_s - Z_s)|^2 \geq 0 \end{cases}$$

So we obtain the expression given by (12) for the generator F and the relation given in Proposition 3 for the optimal strategy ν^* .
It remains to see why ν^* is again in $\mathcal{A}$: from the choice of ν^*, we deduce:

$$|m.(\nu_s^* - (Z_s + \frac{\lambda_s}{\alpha}))| \leq |m.(Z_s + \frac{\lambda_s}{\alpha})|$$

And, hence: $|m.(\nu^* - z)| \leq |m.(Z_s + \frac{\lambda_s}{\alpha})| + |m.|\frac{\lambda_s}{\alpha}||$.
It results then that: $R_s^{\nu^*} = -e^{-\alpha.(x-Y_0)}.\mathcal{E}(-\alpha.(\nu^* - Z)^{'}.M + \alpha.L)$ is a true martingale.

3.3 Power and logarithmic utilities

Similarly as in [8], we introduce two other utility functions and we study for each the corresponding utility maximization problem: in our special case of a general (and, in particular, non Brownian) continuous filtration, we will use the same dynamical method as in the exponential case by introducing a family of random processes and give an expression of both the value function and the optimal strategy.
The main difference with the exponential case is that for those type of utility functions we have to impose furthermore that the wealth process is non negative.
Denoting by U the utility function, the problem we are interested in is to compute $V(x_t)$ defined as in [12]:

$$V(x_t) = ess\sup_{\rho} \mathbb{E}^{\mathcal{F}_t}(U_\gamma(X_T^\rho))$$

where x_t is a fixed $\mathcal{F}_t$-measurable random variable such that for all strategies ρ: $X_t^\rho = x_t$.

In the two cases we will study here, we define another notion of strategy by introducing a d-dimensional process ρ: each component ρ_i stands for the part of the wealth invested in stock i. This will imply we will obtain a very specific expression for the wealth process. Keeping here the same conventions for the price process S as in Section 3.1, we define X^ρ by setting:

$$X_t^\rho = x + \int_0^t X_s^\rho \rho_s^{'}.\frac{dS_s}{S_s} = x + \int_0^t X_s^\rho \rho_s^{'}.dM_s + \int_0^t X_s^\rho.\rho_s^T.d < M >_s \lambda_s$$

The link between the process ν defined in the exponential case and ρ is given by: $\nu = X.\rho$ (where X is the wealth process).
We keep the notation C for the constraint set. Here we impose that the process ρ takes its values in C.

Finally, the wealth process has the following expression:

$$X_t^\rho = x.exp\left(\int_0^t \rho_s^{'}.dM_s - \frac{1}{2}\int_0^t \rho_s^T d < M >_s \rho_s + \int_0^t \rho_s^T.d < M >_s \lambda_s\right)$$
$$= x.\mathcal{E}(\rho^{'}.M)exp(\int_0^t \rho_s^T.d < M >_s \lambda_s)$$

In the sequel, we will study the two following utility functions:
The first one is the power utility:
For all real $\gamma \in]0, 1[$, we will consider: $U_\gamma(x) = \frac{1}{\gamma}.x^\gamma$.
We will fix γ and set: $\mathrm{U}_\gamma = U^1$.
The second one is the logarithmic utility whose definition is: $\mathrm{U}^2(x) = \ln(x)$.
We state hereafter the main results we are able to obtain:

Theorem 3 *Keeping the introduced notation, let V^1 be the value function of the utility maximization problem related to U^1. Its expression is given by:*

$$V^1(x_t) = \frac{x_t^\gamma}{\gamma}.exp(Y_t)$$

and: Y is defined as the unique solution (Y, Z) of a BSDE of the following form:

$$Y_t = 0 - \int_t^T f^1(s, Z_s)dC_s + \int_t^T \frac{1}{2}d < L >_s - \int_t^T Z_s^{'}dM_s - (L_T - L_t)$$

where: L is a $\mathbb{R}$-valued martingale strongly orthogonal to M, and f^1 is given by:

$$f^1(s, z) = \inf_{\rho \in C} \frac{\gamma.(1-\gamma)}{2}\left(|m.(\rho - (\frac{z+\lambda_s}{1-\gamma}))|^2\right) - \frac{\gamma.(1-\gamma)}{2}|m.(\frac{z+\lambda_s}{1-\gamma})|^2 - \frac{1}{2}|m.z|^2 \tag{14}$$

The optimal strategy ρ_1^* is defined for all s ($\mathbb{P}$- a.s) by: $(\rho_1^*)(s) \in \underset{\rho \in C}{argmin}\, |m.(\rho - (\frac{Z_s + \lambda_s}{1-\gamma}))|^2$

Theorem 4 *The expression of the value function V^2 of the utility maximization problem related to U^2 is given by: $V^2(x_t) = \ln(x_t) + Y_t$,*

and: Y is again defined as the solution of the following BSDE (with the terminal condition: $Y_T \equiv 0$):

$$Y_t = 0 - \int_t^T f^2(s, Z_s)dC_s - \int_t^T Z_s^{'}dM_s - \int_t^T dL_s,$$

, where f^2 is given by:

$$f^2(s, z) = \inf_{\rho \in C} \frac{1}{2}|m.(\rho - \lambda_s)|^2 - \frac{1}{2}|m.\lambda_s|^2. \tag{15}$$

The optimal strategy ρ_2^* is defined $\mathbb{P}$-a.s. by: $(\rho_2^*)(s) \in \underset{\rho \in C}{argmin}\, |m.(\rho - \lambda_s)|^2$

Before giving the proofs of these results, we briefly explain the reason why we can claim existence and uniqueness to the BSDEs characterized by (f^1, 0) (resp. by (f^2, 0)).
Remark:
Analogously to the exponential case and before giving the expression of the value function in the two cases, we can show that the two generators (f^1 and f^2) satisfy the estimates given by (H_1) or ($\mathrm{H}_1^{'}$):
On the one hand, the generator f^1 satisfies:

$$-\frac{\gamma.(1-\gamma)}{2}|m.(\frac{z+\lambda_s}{1-\gamma})|^2 - \frac{1}{2}|m.z|^2 \leq f^1(s,z) \leq -\frac{1}{2}|m.z|^2$$

which entails that the generator f^1 satisfies the assumption (H_1).
On the other hand, we claim that: $-\frac{|m.\lambda|^2}{2} \leq f^2(s) \leq 0$,

So f^2 satisfies (H_1).
We check easily here that we have:

$$|f^1(s,z^1) - f^1(s,z^2)| \leq \frac{\gamma}{(1-\gamma)}|m.(z^1-z^2)|(m.|2.\lambda| + |m.z^1| + |m.z^2|)$$

Hence, the condition (H_3) is also satisfied by the generator f^1 which entails that the uniqueness result holds true for the BSDE characterized by (f^1, 0). This condition is again trivially satisfied by f^2 (whose expression does not depend on z).

To prove these theorems, we have to distinguish the two type of utilities because in each case we are going to precise a specific notion of admissible strategies.

3.3.1 Power utility: Proof of Theorem 3

We state below the notion of admissible strategies for the power utility function (this is less restrictive than in the exponential case):

Definition 3 *The set of admissible strategies $\tilde{A}$ consists of all d-dimensional predictable processes $\rho = (\rho_t)$ which satisfy: ρ takes its values in C, and:*
$\int_0^T \rho_s^T d < M >_s \rho_s = \int_0^T |m.\rho_s|^2 dC_s < \infty$, $\mathbb{P}-$ *almost surely.*

This condition entails that the stochastic exponential $\mathcal{E}(\int_0^{\cdot} \rho.dM)$, which is a positive local martingale, is well defined and we write here its expression:

$$\mathcal{E}(\int_0^{\cdot} \rho.dM) = e^{\int_0^{\cdot} \rho_s dM_s - \frac{1}{2}\int_0^{\cdot} \rho_s^T d<M>_s \rho_s}.$$

We recall the definition of the process X^ρ:

$$X_t^\rho = x + \int_0^t X_s^\rho \rho_s^{'} dM_s + \int_0^{\cdot} (m.\rho_s)^T (m.\lambda_s) dC_s$$

Equivalently we obtain the following expression

$$X_{\cdot}^\rho = \mathcal{E}(\rho^{'}.M) e^{\int_0^{\cdot} (m.\rho_s)^T (m.\lambda_s) dC_s}$$

Using the simple computation:

$$(\mathcal{E}(\int_0^{\cdot} \rho^{'} dM))^{\gamma} = \mathcal{E}(\int_0^{\cdot} \gamma.\rho^{'} dM).e^{\frac{\gamma.(\gamma-1)}{2}\int_0^{\cdot} |m.\rho_s|^2 dC_s}$$

it implies that $(X^{\rho})^{\gamma}$ can be written as:

$$(X_t^{\rho})^{\gamma} = \mathcal{E}(\int_0^t \gamma.\rho^{'} dM).e^{\frac{\gamma.(\gamma-1)}{2}\int_0^t |m.\rho_s|^2 dC_s}.e^{\gamma.\int_0^t (m.\rho_s)^T(m.\lambda_s)dC_s}.$$

Remembering here the expression of the BSDE that the process Y satisfies, we have:

$$e^{Y_t} = e^{Y_0}.e^{\int_0^t f^1(s,Z_s)dC_s}\mathcal{E}(\int_0^t Z^{'} dM + L)e^{\frac{1}{2}\int_0^t |m.Z_s|^2 dC_s},$$

where we have used: $\mathcal{E}(\int_0^{\cdot} Z^{'} dM + L) = \mathcal{E}(\int_0^{\cdot} Z^{'} dM).\mathcal{E}(L)$.
We check easily:

$$\mathcal{E}(\int_0^{\cdot} (\gamma.\rho + Z)^{'} dM) = \mathcal{E}(\int_0^{\cdot} \gamma.\rho^{'} dM).\mathcal{E}(\int_0^{\cdot} Z^{'} dM).e^{-\int_0^{\cdot} \gamma.(m.\rho)^T(m.Z)dC_s}$$

From these equalities, we derive the following expression for:
$R_t^{\rho} = \frac{1}{\gamma}(X_t^{\rho})^{\gamma}.e^{Y_t}$:

$$R_t^{\rho} = \frac{1}{\gamma}\mathcal{E}(\int_0^t (\gamma.\rho + Z)^{'} dM + L_t)e^{\tilde{A}_t^{\rho}}.$$

where the process $\tilde{A}^{\rho}$ is given by the formula :

$$\tilde{A}_t^{\rho} = \int_0^t \big(f^1(s,Z_s) + \frac{1}{2}|m.Z_s|^2 + \frac{\gamma.(\gamma-1)}{2}|m.\rho|^2 + \gamma.(m.\rho)^T(m.(Z_s + \lambda))\big)dC_s$$

Thanks to the assumptions of admissibility of ρ and provided we obtain a BSDE with quadratic growth (we will have controls of the BMO norms of the process $Z^{'}$.M and L), we can claim that: $\mathcal{E}(\int_0^t (\gamma.\rho + Z)^{'} dM + L_t)$ is a positive local martingale.

The choice of f^1 such that: $\tilde{A}^{\rho} \leq 0$, implies that there exists a family of $\mathbb{F}$-stopping time (τ_k) such that, for each k, $(R_{t\wedge\tau_k}^{\rho})$ is a super martingale.
Since R^{ρ} is bounded from below by 0, passing to the limit when k $\to \infty$ in the following inequality:

$$\forall\ A \in \mathcal{F}_s \quad \mathbb{E}^{\mathcal{F}_s}\big(R_{t\wedge\tau_k}^{\rho}.\mathbf{1}_A\big) \leq R_{s\wedge\tau_k^{\rho}}.\mathbf{1}_A$$

we obtain that R^{ρ} is a super martingale, for all admissible strategies ρ.
In the same way that in the proof in the exponential case, we can show that R^{ρ^*} is a martingale and ρ^* is admissible.

3.3.2 Logarithmic utility: Proof of Theorem 4

Definition 4 *The set of admissible strategies $\tilde{A}$ consists of all d-dimensional predictable processes $\rho = (\rho_t)$ which satisfy: ρ takes its values in C, and:*
$\mathbb{E}\big(\int_0^T \rho_s^T d < M >_s \rho_s\big) < \infty$.

Since we keep the same expression for the wealth process, we have:

$$\ln(X_t^\rho) = \ln(x) + \int_0^t \rho_s dM_s - \frac{1}{2}.\int_0^t |m.\rho_s|^2 dC_s + \int_0^t (m.\rho_s)^T (m.\lambda_s) dC_s$$

In this case, Y is solution of a BSDE of the following type:

$$Y_t = 0 - \int_t^T f^2(s, Z_s) dC_s - \int_t^T Z_s^{'} dM_s - \int_t^T dL_s$$

Writing the process $(\ln(X^\rho)$ $+Y)$ under the following form:

$$ln(X_t^\rho) + Y_t = ln(x) + Y_0 + \int_0^t \big((\rho + Z)^{'} dM_s + dL_s\big) + A^2(t)$$

The process A^2 is given by:

$$A^2(t) = \int_0^t (f^2(s, Z_s) - \frac{1}{2}.|m.\rho_s|^2 + (m.\rho_s)^T (m.\lambda_s)) dC_s$$

We conclude easily that $\ln(X^\rho) + Y$ is a super martingale for any $\rho \in \tilde{A}$ and $\ln(X^{\rho^*}) + Y$ is a martingale, thanks to the choice of f^2, whose expression is given by (15) .

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